Mathematics Analysis and Approaches HL
Internal Assessment

Application of the shortest path problem

Number of pages: 12

Introduction

Some time ago I took a trip with Uber and during that trip I was wondering if they took me for the shortest path. So, in this exploration my aim is to try to figure out if they took me for the shortest route. The shortest path problem is a well-known problem in maths, more precisely in Graph theory, which I will explain later. There are different algorithms in this theory, but I chose to use the Dijkstra's algorithm. Figure 1 is a screenshot from the Uber site which shows the route they took me, that took 5.8km according to their website.

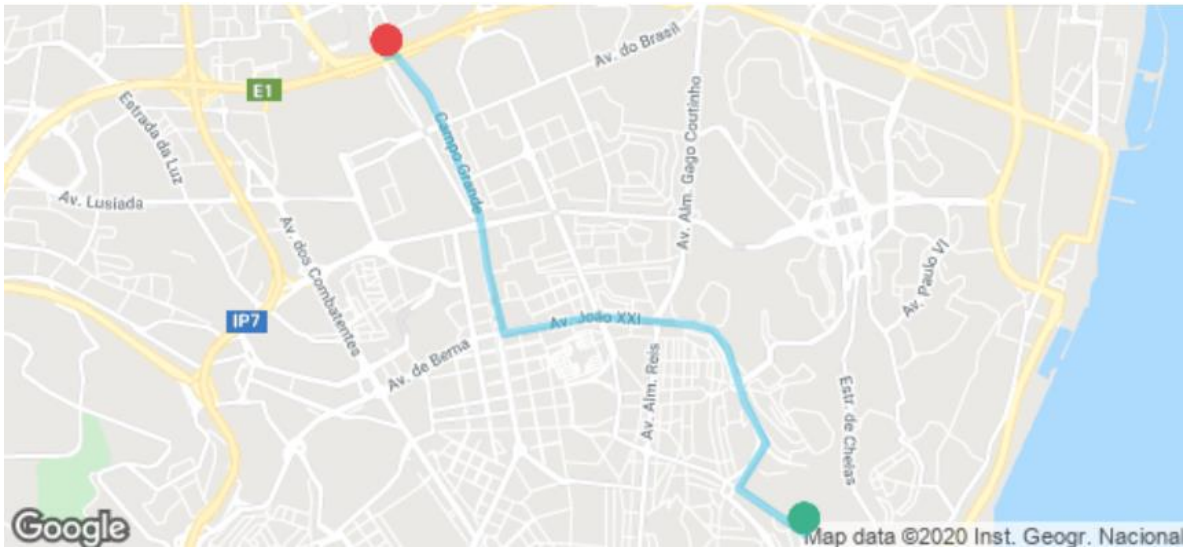

Figure 1 – Uber trip

Graph Theory

In 1736 Euler published the first paper about graph theory, even if at the time he did not call it graph theory, this paper was the beginning of a new field in mathematics. At that time there was a problem that divided mathematicians, the Königsberg bridge problem. This problem asks if it was possible to cross the seven bridges of Königsberg (at that time Germany but now is part of Russia) which pass over the river Preger in a single trip. But there are two conditions, the first one you cannot pass two times the same bridge. And the second that you must end in the same point which you began. In Figure 2 you have the image of the city at that time and its bridges.

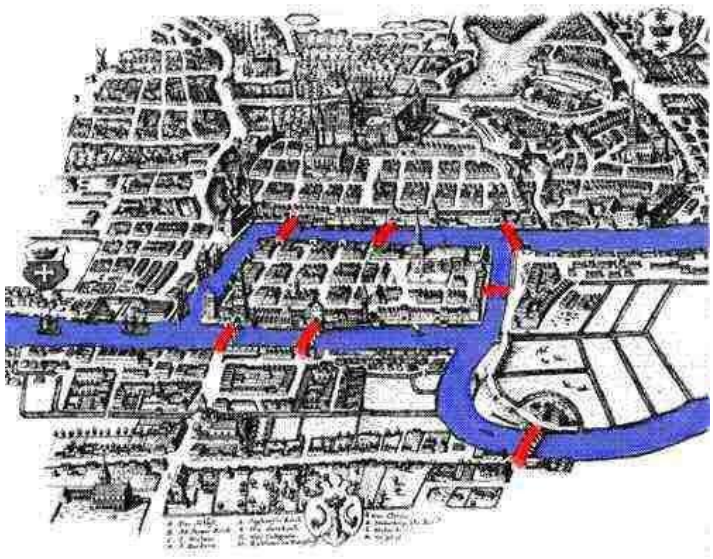

Figure 2 - Königsberg bridges

As I said that problem divided the mathematicians at that time with some arguing it was possible and others arguing otherwise. But in 1736 Euler proved that it was not possible in that paper, which set the beginning of graph theory.

Figure 2 - *Leonard Euler's Solution to the Konigsberg Bridge Problem | Mathematical Association of America*. https://www.maa.org/press/periodicals/convergence/leonard-eulers-solution-to-the-konigsberg-bridge-problem. Accessed 24 Nov. 2020.

In this theory the term graph has a different connotation from the one we are used to as data charts. In graph theory, graphs refer to a set of vertices and edges that connect the vertices. There are different types of graphs. There are simples' graphs where one edge connects two vertices and there are no loops. There are also multigraphs where two vertices are joint by more than one edge, which are the graphs that I am going to use to solve my problem. Moreover, for my problem I will need to use weighted graphs i.e., graphs where its edges have associated values to them. In this case the values will represent the distance from each vertex to the other. In Figure 3 there is an example of a weighted multigraph.

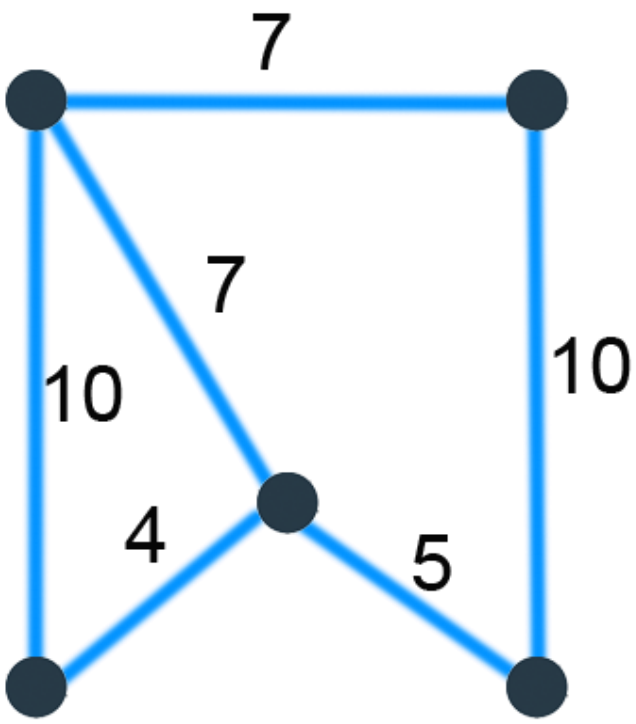

Figure 3 – Weighted multigraph

Shortest path problem and Dijkstra's Algorithm

In graph theory there is a well-known problem called the shortest path problem, where you want to find a path between two vertices such that the sum of the weights is minimized.

Edsger Djikstra, a computer scientist in the 20th century was of extreme help in the field of graph theory. In 1956 he created an algorithm which could solve the shortest path problem. This is a simple algorithm but very powerful to solve this problem.

Before applying the algorithm, we need to assume that we are only moving forward, for example in Figure 4 we are going from a to h and therefore we will not count paths moving backwards. Like if we move from a to b to c to d and then to e we will not check back to b. In Figure 4 all values are expressed in meters.

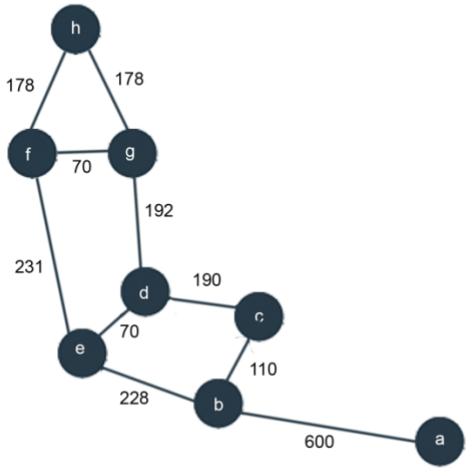

Figure 4 – Simple graph

The idea of this theorem is to start at point a and visit the points that are connected to it, in this case only b. Then visit the points connected to b, in this case c and e. After that we will continue with the smallest known distance vertex and see the smallest distances. And go on like this until we reach point j. It is easier to understand after seeing how it applies to a specific problem, thus below it is shown how the algorithm applies.

Starting in point a, we updated the table with a value of zero in the vertex a (our initial point) and a value of “infinite” to the other vertexes, as it is the shortest cost until now (Table 1).

Vertex	Shortest distance from a (m)	Previous vertex
a	0	
b	∞	
c	∞	
d	∞	
e	∞	
f	∞	
g	∞	
h	∞	

Table 1

Then the closest point to a is point b, with a cost of 600m. Therefore we will update it in our table by cutting the “infinite” in vertex b and switching it for the shortest value known until now. Also we need to update the collum of previous vertex in row b to a.

Vertex	Shortest distance from a (m)	Previous vertex
a	0	
b	~~∞~~ ; 600	a
c	∞	
d	∞	
e	∞	
f	∞	
g	∞	
h	∞	

Table 2

After that we will start in vertex b and check again the shortest path. In this case we have two options. We can go either to c or e. So we will sum the distance from a to b, which is 600 to the individual distances.

$$Distance\ to\ vertex\ c = 600 + 110 = 710m$$

$$Distance\ to\ vertex\ e = 600 + 228 = 828m$$

Now we have two vertices to update on our table, the vertex c and e. We will do the same we did for vertex b, as it shown in Table 3.

Vertex	Shortest distance from a (m)	Previous vertex
a	0	
b	~~∞~~ ; 600	a
c	~~∞~~; 710	b
d	∞	
e	~~∞~~; 828	b
f	∞	
g	∞	
h	∞	

Table 3

Now the work will we have to do is the same. We must visit each vertex and write down in the table the shortest distance known at the moment. In table 4 you can see the final table when all vertices have been visited.

Vertex	Shortest distance from a	Previous vertex
a	0	
b	~~∞~~ ; 600	a
c	~~∞~~ ; 710	b
d	~~∞~~ ; ~~900~~ ; 898	~~c~~ ; e
e	~~∞~~ ; 828	b
f	~~∞~~ ; 1059	e
g	~~∞~~ ; 1090	d
h	~~∞~~ ; 1237	f

Table 4

After all this work, we get that the shortest path from a to h goes by : a → b → e → f → h, and has a total cost of 1237 m.

Applying the theorem to my problem

After seeing that the Dijkstra's theorem applies to a simpler problem, now we can apply it to a more complex one. This algorithm is very helpful but it is very difficult to do by hand when we have a problem with a lot of vertices, like my Uber trip. Therefore, I will use a software that runs the Djikstra's algorithm to help me solve the shortest path for the Uber trip.

The first step to solve the problem is to draw a simpler version of the map. In this version I will exclude some neighbourhoods, as we know without the need of a calculation, that going in circles around a neighbourhood will take longer. In Figure 5 we have the image from google maps and in Figure 6 the simpler representation of the map. Also, we will need to make the same assumption we did in the example, we are only moving forward, we will not count paths moving backwards.

Figure 5 - Map

Figure 6 – Representation of the map

The next step is to construct the graph. At each intersection between two or more roads, we will set a vertex. And each line that connect two vertices will be a weighted edge where the edges will have the distance associated. I chose to do two separate figures to show the vertices and the edges. If they were only in one figure it would be a mess, therefore in two different figure it is much easier to understand. In Figure 7 we have the representation with vertices. And in Figure 8 we have the representation of the edges where all values are in meters.

Figure 7 – Representation of the graph with the name of the vertices

Figure 8 – Representation of the graph with weighted edges

In Figure 8, I got the values using a tool in google maps to measure distances. The problem with this tool is that the values are only an approximation and the error can be high, as I got the values by setting the initial and final point by the eye.

Now that we got the graphs we can put them in the algorithm to solve the shortest path. I used an online programme created in JavaScript which runs the Dijkstra's algorithm.

After inserting all the vertexes and weighted edges, the result I got for the shortest path is:

$$1 \rightarrow 2 \rightarrow 3 \rightarrow 5 \rightarrow 7 \rightarrow 44 \rightarrow 45 \rightarrow 51 \rightarrow 52 \rightarrow 53 \rightarrow 54 \rightarrow 88 \rightarrow 89 \rightarrow 90 \rightarrow 119 \rightarrow 120 \rightarrow 121 \rightarrow 136 \rightarrow 137 \rightarrow 138 \rightarrow 153 \rightarrow 154 \rightarrow 159 \rightarrow 160 \rightarrow 161 \rightarrow 162 \rightarrow 163 \rightarrow 174$$

And the value for the total distance is 5345m. Figure 9 shows the shortest path.

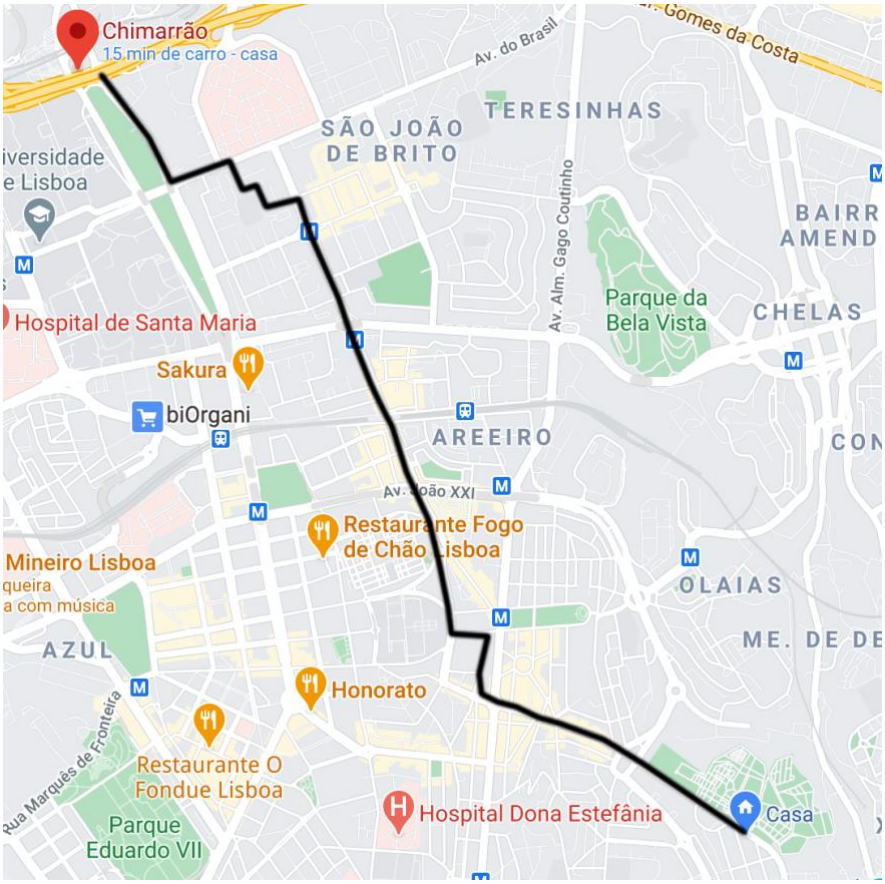

Figure 9 – Shortest path

Conclusion

The objective of this exploration was to find out if Uber took me for the shortest path or not. Which we have seen that did not, Uber took me for a 455m longer trip. This does not mean that Uber is cheating me. There are some other factors to consider. The first one, we must not forget that the values I used in my calculations are a simple approximation of the real values and have some error associated. Then Uber takes more into account than only the shortest path. Like traffic lights and traffic.

Therefore, we can say that the shortest path is not the one Uber took me but this does not mean that Uber is cheating me.

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